More properties of almost Cohen-Macaulay rings

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Abstract

Some interesting properties of almost Cohen-Macaulay rings are investigated and a Serre type property connected with this class of rings is studied.

1 Introduction

A flaw in the chapter dedicated to Cohen-Macaulay rings in the first edition of [5] was corrected in the second edition. This led to the study of the so-called almost Cohen Macaulay rings, first by Y. Han [1] and later by M.-C. Kang [2], [3]. Since the first of these papers is written in Chinese, the others two are the main reference for the subject.

Remark 1.1 *Let A be a commutative noetherian ring, $P \in \mathrm{Spec}(A)$ and $M \neq 0$ a finitely generated A-module. Then* $\mathrm{depth}_P(M) \leq \mathrm{depth}_{PA_P} M_P$.

Definition 1.2 (cf. [1], [2]) *Let A be a commutative noetherian ring. A finitely generated A-module $M \neq 0$ is called almost Cohen-Macaulay if* $\mathrm{depth}_P M = \mathrm{depth}_{PA_P} M_P$, *for any $P \in \mathrm{Supp}(M)$. A is called an almost Cohen-Macaulay ring if it is an almost Cohen-Macaulay A-module, that is if for any $P \in \mathrm{Spec}(A)$,* $\mathrm{depth}_P A = \mathrm{depth}_{PA_P} A_P$.

Several properties of almost Cohen-Macaulay rings are proved in [2] and several interesting examples are given in [3]. In the following we are trying to complete the results in [2] and to introduce a Serre-type condition, that we call (C_k), for any $k \in \mathbb{N}$, condition that is to be to almost Cohen-Macaulay rings what the classical Serre condition (S_k) is to Cohen-Macaulay rings.

2 Properties of almost Cohen-Macaulay rings

All rings considered will be commutative and with unit. We start by reminding some basic properties of almost Cohen-Macaulay rings.

Remark 2.1 *Let A be a noetherian ring. Then:*
a) A is almost Cohen-Macaulay iff $\mathrm{ht}(P) \leq 1 + \mathrm{depth}_P A, \forall\ P \in \mathrm{Spec}(A)$ ([2], 1.5)*;*
b) A is almost Cohen-Macaulay iff A_P is almost Cohen-Macaulay for any $P \in \mathrm{Spec}(A)$ iff A_Q is almost Cohen-Macaulay for any $Q \in \mathrm{Max}(A)$ iff $\mathrm{ht}(Q) \leq 1 + \mathrm{depth} A_Q$ for any $Q \in \mathrm{Max}(A)$ ([2], 2.6)*;*
c) If A is local, it follows from b) that A is almost Cohen-Macaulay if and only if $\dim(A) \leq 1 + \mathrm{depth}(A)$.

Our first result is a stronger formulation of [2], 2.10 and deals with the behaviour of almost Cohen-Macaulay rings with respect to flat morphisms.

Proposition 2.2 *Let $u : (A, m) \to (B, n)$ be a local flat morphism of noetherian local rings.*
a) If B is almost Cohen-Macaulay, then A and B/mB are almost Cohen-Macaulay.
b)If A and B/mB are almost Cohen-Macaulay and one of them is Cohen-Macaulay, then B is almost Cohen-Macaulay.

Proof: a) We have

$$\dim(A) = \dim(B) - \dim(B/mB) \leq 1 + \mathrm{depth} B - \dim(B/mB) \leq$$

$$\leq 1 + \mathrm{depth} B - \mathrm{depth}(B/mB) = 1 + \mathrm{depth} A.$$

We have also

$$\dim(B/mB) - \mathrm{depth}(B/mB) = (\dim(B) - \mathrm{depth} B) - (\dim(A) - \mathrm{depth} A) \leq$$

$$\leq 1 - (\dim(A) - \mathrm{depth} A) \leq 1.$$

b) Since u is flat we have

$$\dim(B) = \dim(A) + \dim(B/mB) \leq 1 + \mathrm{depth}(A) + \mathrm{depth}(B/mB) =$$

$$= 1 + \mathrm{depth}(B).$$

Question 2.3 *We don't know of any example of a local flat morphism of noetherian local rings $u : (A, m) \to (B, n)$ such that A and B/mB are almost Cohen-Macaulay and B is not almost Cohen-Macaulay.*

Corollary 2.4 *Let A be a noetherian local ring, $I \neq A$ be an ideal contained in the Jacobson radical of A and $\hat{A}$ the completion of A in the I-adic topology. Then A is almost Cohen-Macaulay if and only if $\hat{A}$ is almost Cohen-Macaulay.*

Proof: Since I is contained in the Jacobson radical of A, the canonical morphism $A \to \hat{A}$ is faithfully flat and $\mathrm{Max}(A) \cong \mathrm{Max}(\hat{A})$. Moreover, if $m \in \mathrm{Max}(A)$ and $\hat{m}$ is the corresponding maximal ideal of $\hat{A}$, the closed fiber of the morphism $A_m \to \hat{A}_{\hat{m}}$ is a field. Now apply 2.2.

Corollary 2.5 (see [2], 1.6) *Let A be a noetherian ring and $n \in \mathbb{N}$. Then A is almost Cohen-Macaulay if and only if $A[[X_1, \ldots, X_n]]$ is almost Cohen-Maculay.*

Proof: Suppose that A is almost Cohen-Macaulay. We may clearly assume that A is local and $n = 1$. By [2], 1.3 we get that $A[X]_{(X)}$ is almost Cohen-Macaulay. Now apply 2.4. The converse is clear.

For the next corollary we need some notations.

Notation 2.6 *If $\boldsymbol{P}$ is a property of noetherian local rings, we denote by $\boldsymbol{P}(A) := \{Q \in \mathrm{Spec}(A) \mid A_Q$ has the property $\boldsymbol{P}\}$ and by $\boldsymbol{NP}(A) := \{Q \in \mathrm{Spec}(A) \mid A_Q$ has not the property $\boldsymbol{P}\} = \mathrm{Spec}(A) \setminus \boldsymbol{P}(A)$.*

Definition 2.7 *Let A be a noetherian ring. According to 2.6, the set*

$$\mathbf{aCM}(A) := \{P \in \mathrm{Spec}(A) \mid A_P \textit{ is almost Cohen-Macaulay}\}$$

is called the almost Cohen-Macaulay locus of A.

Corollary 2.8 *Let $u : A \to B$ be a morphism of noetherian local rings and $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the induced morphism on the spectra. If the fibers of u are Cohen-Macaulay, then $\varphi^{-1}(\mathbf{aCM}(A)) = \mathbf{aCM}(B)$.*

Proof: Obvious from 2.2.

In Cohen-Macaulay rings chains of prime ideals behave very well, in the sense that Cohen-Macaulay rings are universally catenary(see [5]). This is no more the case for almost Cohen-Macaulay rings.

Example 2.9 *There exists a local almost Cohen-Macaulay ring which is not catenary.*

Proof: Indeed, by [2], Ex. 2, any noetherian normal integral domain of dimension 3 is almost Cohen-Macaulay. In [6] such a ring which is not catenary is constructed.

The next result shows that some of the formal fibres of almost Cohen-Macaulay rings are almost Cohen-Macaulay. A stronger fact will be proved in 2.13.

Proposition 2.10 *Let A be a noetherian local almost Cohen-Macaulay ring, $P \in \mathrm{Spec}(A), Q \in \mathrm{Ass}(\hat{A}/P\hat{A})$. Then $\hat{A}_Q/P\hat{A}_Q$ is almost Cohen-Macaulay.*

Proof: We have

$$\dim(\hat{A}_Q/P\hat{A}_Q) = \dim \hat{A}_Q - \dim A_P \leq \mathrm{depth}\hat{A}_Q + 1 - \dim A_P \leq$$

$$\leq \mathrm{depth}\hat{A}_P + 1 - \dim A_P = \mathrm{depth}(\hat{A}_Q/P\hat{A}_Q) + 1.$$

The following result shows that the almost Cohen-Macaulay property is preserved by tensor products and finite field extensions.

Proposition 2.11 *Let k be a field, A and B be two $k-$algebras such that $A \otimes_k B$ is a noetherian ring. If A and B are almost Cohen-Macaulay, then $A \otimes_k B$ is almost Cohen-Macaulay.*

Proof: Let $P \in \mathrm{Spec}(A)$. We have a flat morphism $B \to B \otimes_k k(P)$ and let $Q \in \mathrm{Spec}(B)$. Set $T := A/P \otimes_k B/Q = A \otimes_k B/(P \otimes_k B + A \otimes_k Q)$. Then $k(P) \otimes_k k(Q)$ is a ring of fractions of T, hence noetherian by assumption. By [7], 1.5, it follows that $k(P) \otimes_k k(Q)$ is locally a complete intersection. Let now $Q \in \mathrm{Spec}(B)$ and $P = Q \cap A$. By the above the flat local morphism $A_P \to (B \otimes_k k(P))_Q$ has a complete intersection closed fiber, hence the ring $(B \otimes_k k(P))_Q$ is almost Cohen-Macaulay by 2.2. Now consider the flat morphism $A \to A \otimes_k B$, let $Q \in \mathrm{Spec}(A \otimes_k B)$ and $P = Q \cap A$. Then the flat local morphism $A_P \to (A \otimes_k B)_Q$ has a complete intersection closed fiber, whence $(A \otimes_k B)_Q$ is almost Cohen-Macaulay.

Corollary 2.12 *Let k be a field, A a noetherian $k-$algebra which is almost Cohen-Macaulay and L a finite field extension of k. Then $A \otimes_k L$ is almost Cohen-Macaulay.*

As for the Cohen-Macaulay property, the formal fibres of factorizations of almost Cohen-Macaulay rings are almost Cohen-Macaulay.

Proposition 2.13 *Let B be a local almost Cohen-Macaulay ring, I an ideal of B and $A = B/I$. Then the formal fibers of A are almost Cohen-Macaulay.*

Proof: We have $\hat{A} = \hat{B} \otimes_B A = \hat{B}/I\hat{B}$, hence the formal fibers of A are exactly the formal fibers of B in the prime ideals of B containing I. Let P be such a prime ideal, let $S = B \setminus P$ and let $C := S^{-1}(\hat{B}/I\hat{B})$. Let also $Q \in \mathrm{Spec}(C)$. There exists $Q' \in \mathrm{Spec}(\hat{B})$ such that $Q = Q'C$ and $Q' \cap B = P$. Thus we have a local flat morphism $B_Q \to \hat{B}_{Q'}$. But B is almost Cohen-Macaulay, hence $\hat{B}_{Q'}$ and consequently $C_Q \cong \hat{B}_{Q'}/P\hat{B}_{Q'}$ are almost Cohen-Macaulay, by 2.2.

3 The property (C_n)

Recall that given a natural number n, a noetherian ring A is said to have Serre property (S_n) if $\mathrm{depth}(A_P) \geq \min(\mathrm{ht}P, n)$ for any prime ideal $P \in \mathrm{Spec}(A)$. Moreover, A is Cohen-Macaulay if and only if A has the property (S_n) for any $n \in \mathbb{N}$ (see [5], (17.I)). We will try to characterize almost Cohen-Macaulay rings in a similar way.

Definition 3.1 *Let $n \in \mathbb{N}$ be a natural number. We say that a noetherian ring A has the property (C_n) if* $\mathrm{depth}(A_P) \geq \min(\mathrm{ht}P, n) - 1, \forall\ P \in \mathrm{Spec}(A)$.

Remark 3.2 *a) It is clear that $(C_n) \Rightarrow (C_{n-1})$ and that $(S_n) \Rightarrow (C_n), \forall n \in \mathbb{N}$.*
b) It is also clear that if A has (C_n), then A_P has $(C_n), \forall P \in \mathrm{Spec}(A)$.

Theorem 3.3 *A noetherian ring A is almost Cohen-Macaulay if and only if A has the property (C_n) for every $n \in \mathbb{N}$.*

Proof: Assume that A is almost Cohen-Macaulay and let $P \in \operatorname{Spec}(A)$. Then A_P is almost Cohen-Macaulay, hence $\operatorname{depth}(A_P) \geq \operatorname{ht}(P)-1$. If $n \geq \operatorname{ht}(P)$, then $\min(\operatorname{ht}(P), n) = \operatorname{ht}(P)$, hence $\operatorname{depth}(A_P) \geq \min(n, \operatorname{ht}(P))-1$. If $n < \operatorname{ht}(P)$, then $\min(n, \operatorname{ht}(P)) = n$, so that $\operatorname{depth}(A_P) \geq \operatorname{ht}(P)-1 > n-1 = \min(\operatorname{ht}(P), n)-1$.
For the converse, let $P \in \operatorname{Spec}(A), \operatorname{ht}(P) = l$. Then

$$\operatorname{depth}(A_P) \geq \min(l, \operatorname{ht}(P)) - 1 = \operatorname{ht}(P) - 1.$$

Proposition 3.4 *Let $k \in \mathbb{N}$. A noetherian ring A has the property (C_k) if and only if A_P is almost Cohen-Macaulay for any $P \in \operatorname{Spec}(A)$ with $\operatorname{depth}(A_P) \leq k-2$.*

Proof: Let $P \in \operatorname{Spec}(A)$ such that $\min(k, \operatorname{ht}(P)) - 1 \leq \operatorname{depth}(A_P) \leq k-2$. If $\operatorname{ht}(P) \leq k$, then $\operatorname{depth}(A_P) \geq \operatorname{ht}(P) - 1$. And if $\operatorname{ht}(P) > k$, then it follows that $k-2 > \operatorname{depth}(A_P) \geq k-1$. Contradiction!
Conversely, let $P \in \operatorname{Spec}(A)$. If $\operatorname{depth}(A_P) \leq k-2$, then A_P is almost Cohen-Macaulay, hence $\operatorname{ht}(P) - 1 \leq \operatorname{depth}(A_P) \leq k-2$. Thus $\min(\operatorname{ht}(P), k) = \operatorname{ht}(P)$, whence $\operatorname{depth}(A_P) \geq \min(k, \operatorname{ht}(P)$. If $k-2 < \operatorname{depth}(A_P)$, then $\operatorname{ht}(P) > k-2$, hence $\operatorname{depth}(A_P) \geq \min(k, \operatorname{ht}(P)) - 1$.

Proposition 3.5 *Let A be a noetherian ring, $k \in \mathbb{N}$ and $x \in A$ a non zero divisor. If A/xA has the property (C_k), then A has the property (C_k).*

Proof: Let $Q \in \operatorname{Spec}(A)$ such that $\operatorname{depth}(A_Q) = n \leq k-2$. If $x \in Q$, then $\operatorname{depth}(A/xA)_Q = n-1 \leq k-3$. Then $\operatorname{ht}(Q/xA) \leq n-1+1 = n$, hence $\operatorname{ht}(Q) \leq n+1 = \operatorname{depth} A_Q + 1$. If $x \notin Q$, let $P \in \operatorname{Min}(Q+xA)$. Then $(P+xA)A_Q$ is QA_Q-primary and $\operatorname{depth}(A_P) \leq \operatorname{depth}(A_Q)+1 = n+1$. Then $\operatorname{depth}(A/xA)_Q = n-1$, hence $\operatorname{ht}(P/xA) \leq n$. It follows that $\operatorname{ht}(P) \leq n+1 = \operatorname{depth}(A_P) + 1$.

Definition 3.6 *We say that a property $\boldsymbol{P}$ of noetherian local rings satisfies Nagata's Criterion (NC) if the following holds: if A is a noetherian ring such for every $P \in \boldsymbol{P}(A)$, the set $\boldsymbol{P}(A/P)$ contains a non-empty open set of $\operatorname{Spec}(A/P)$, then $\boldsymbol{P}(A)$ is open in $\operatorname{Spec}(A)$.*

An interesting study of Nagata Criterion is performed in [4].

Theorem 3.7 *Let $k \in \mathbb{N}$. The property (C_k) satisfies (NC).*

Proof: Let $Q \in C_k(A)$. Then $\operatorname{depth}(A_Q) \geq \min(k, \operatorname{ht}(Q)) - 1$.
Case a): $\operatorname{ht}(Q) \leq k$. Then $\min(k, \operatorname{ht}(Q)) = \operatorname{ht}(Q)$, hence $\operatorname{depth}(A_Q)+1 \geq \operatorname{ht}(Q)$ and A_Q is almost Cohen-Macaulay. Let $f \in A \setminus Q$ such that

$$\dim(A_P) = \dim(A_Q) + \dim(A_P/QA_P)$$

and

$$\operatorname{depth}(A_P) = \operatorname{depth}(A_Q) + \operatorname{depth}(A_P/QA_P)$$

for any $P \in D(f) \cap V(Q) \cap NT_k(A)$. Then $\operatorname{depth}(A_P) \ngeq \min(k, \operatorname{ht}(P)) - 1$.

Case a1): $\mathrm{ht}(P) \leq k$. Then $\min(k, \mathrm{ht}(P)) = \mathrm{ht}(P)$, hence $\mathrm{depth}(A_P) + 1 < \mathrm{ht}(P)$. Then

$$\mathrm{depth}(A_P/QA_P) + 1 = \mathrm{depth}(A_P) - \mathrm{depth}(A_Q) + 1 <$$

$$< \mathrm{ht}(P) - \mathrm{depth}(A_Q) \leq \mathrm{ht}(P) - \mathrm{ht}(Q) + 1.$$

Then $\mathrm{depth}(A_P/QA_P) < \dim(A_P/QA_P) = \dim(A_P) - \dim(A_Q)$ and it follows that A_P/QA_P is not (C_k).
Case a2): $\mathrm{ht}(P) > k$. Then $\min(k, \mathrm{ht}(P)) = k$, hence $\mathrm{depth}(A_P) < k - 1$. It follows that

$$\mathrm{depth}(A_P/QA_P) = \mathrm{depth}(A_P) - \mathrm{depth}(A_Q) <$$

$$< k - 1 + 1 - \mathrm{ht}(Q) = k - \mathrm{ht}(Q).$$

This implies that A_P/QA_P is not (C_k).
Case b): $\mathrm{ht}(Q) > k$. Then $\min(k, \mathrm{ht}(Q)) = k$ and $\mathrm{depth}(A_Q) + 1 \geq k$. Since $\mathrm{ht}(P) > k$, it follows that $\min(k, \mathrm{ht}(P)) = k$ and $\mathrm{depth}(A_P) + 1 < k$. Let $x_1, \ldots, x_r$ be an A_Q-regular sequence. Then there exists $f \in A \setminus Q$ such that $x_1, \ldots, x_r$ is A_f-regular. If $P \in D(f) \cap V(Q)$, it follows that A_P is (C_k).

Corollary 3.8 *The property almost Cohen-Macaulay satisfies (NC).*

Theorem 3.9 *Let A be a quasi-excellent ring and $k \in \mathbb{N}$. Then $C_k(A)$ and* $\mathbf{aCM}(A)$ *are open in the Zariski topology of* $\mathrm{Spec}(A)$.

Proof: Let $P \in \mathrm{Spec}(A)$. Then $\mathbf{aCM}(A/P)$ and $C_k(A/P)$ contain the non-empty open set $\mathbf{Reg}(A/P) = \{P \in \mathrm{Spec}(A) \mid A_P \text{ is regular }\}$. Now apply 3.7 and 3.8.

Corollary 3.10 *Let A be a complete semilocal ring and $k \in \mathbb{N}$. Then $C_k(A)$ and $aCM(A)$ are open in the Zariski topology of* $\mathrm{Spec}(A)$.

Corollary 3.11 *Let A be a noetherian local ring with Cohen-Macaulay formal fibers. Then* $\mathbf{aCM}(A)$ *is open.*

Proof: Follows from 3.10 and 2.8.

Proposition 3.12 *Let $u : A \to B$ be a flat morphism of noetherian rings and $k \in \mathbb{N}$. If B has (C_k), then A has (C_k).*

Proof: We may assume that A and B are local rings and that u is local. Let $P \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Min}(PB)$. Then $\dim(B_Q/PB_Q) = 0$, hence

$$\mathrm{depth}(A_P) = \mathrm{depth}(B_Q) \geq \min(k, \dim(B_Q)) - 1 =$$

$$= \min(k, \dim(A_P)) - 1.$$

Proposition 3.13 *Let $u : A \to B$ be a flat morphism of noetherian rings and $k \in \mathbb{N}$.*
a) If A has (C_k) and all the fibers of u have (S_k), then B has (C_k).
b) If A has (S_k) and all the fibers of u have (C_k), then B has (C_k).

Proof: a) Let $Q \in \mathrm{Spec}(B), P = Q \cap A$. Then by flatness we have

$$\dim(B_Q) = \dim(A_P) + \dim(B_Q/PB_Q),$$

$$\mathrm{depth}(B_Q) = \mathrm{depth}(A_P) + \mathrm{depth}(B_Q/PB_Q).$$

By assumption we have

$$\mathrm{depth}(A_P) \geq \min(k, \mathrm{ht}(P)) - 1,$$

$$\mathrm{depth}(B_Q/PB_Q) \geq \min(k, \dim(B_Q/PB_Q).$$

Hence we have

$$\mathrm{depth}(B_Q) = \mathrm{depth}(A_P) + \mathrm{depth}(B_Q/PB_Q) \geq$$

$$\geq \min(k, \mathrm{ht}(P)) - 1 + \min(k, \dim(B_Q/PB_Q)) = \min(k, \mathrm{ht}(B_Q)) - 1.$$

b) The proof is the same.
As a corollary we get a new proof of a previous result.

Corollary 3.14 *Let $u : A \to B$ be a flat morphism of noetherian rings.*
a) If B is almost Cohen-Macaulay, then A is almost Cohen-Macaulay.
b) If A is almost Cohen-Macaulay and the fibers of u are Cohen-Macaulay, then B is almost Cohen-Macaulay.

Example 3.15 *Let k be a field and let X_0, X_1, X_2, Y_1, Y_2 be indeterminates. Set $B = k[[X_0, X_1, X_2]]/(X_0) \cap (X_0, X_1)^2 \cap (X_0, X_1, X_2)^3$ and $A := B[[Y_1, Y_2]]$. It is easy to see that A is a noetherian local ring with $\dim(A) = 5, \mathrm{depth}(A) = 2$. It is also not difficult to see that A has the property (C_3) and not the property (C_4). Similar other examples can easily be constructed.*

Example 3.16 *Let k be a field, X, Y indeterminates and consider the ring $A = k[[X, Y]]/(X^2, XY)$. Then A has (C_2) and not (S_2).*

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